

Integral positive (negative) quandle cocycle invariants are trivial for knots

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Abstract In this note we prove that for any finite quandle X and any 2-cocycle $\phi \in Z_{Q\pm}^2(X; \mathbb{Z})$, the cocycle invariant $\Phi_{\phi}^{\pm}(K)$ is trivial for all knots K .

Keywords quandle, 2-cocycle invariant

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1 Introduction

A quandle [9, 12] is a set with a binary operation which satisfies some axioms motivated by variations on Reidemeister moves. Similar to the knot group, for every knot $K \subset S^3$ one can define the knot quandle $Q(K)$ [9]. It is well known that $Q(K)$ is a powerful knot invariant. It characterizes the knot K up to inversion. However in general the knot quandle is not easy to deal with, therefore it is more convenient to count the homomorphisms from $Q(K)$ to a fixed finite quandle, known as the quandle coloring invariant. More sophisticated invariants of this sort were introduced via quandle cohomology in [3]. More precisely, as a modification of the rack cohomology theory [6, 7], J. S. Carter et al constructed the quandle cohomology theory, then they showed that with a given quandle 2-cocycle (3-cocycle) one can generalize the quandle coloring invariant to a state-sum invariant for knots (knotted surfaces). In particular when the 2-cocycle is a coboundary this state-sum invariant reduces to the quandle coloring invariant. In [4] we consider another boundary map of the chain complex and introduce the positive quandle cohomology theory. Similar to the quandle cohomology theory, with a given positive quandle 2-cocycle and 3-cocycle we can also define a state-sum invariant for knots and knotted surfaces respectively. Besides of the trivial quandle, it was proved that if we choose the dihedral quandle then the quandle cocycle invariants and positive quandle cocycle invariants are both trivial (i.e. reduce to the quandle coloring invariant) for knots [3, 4]. The main result of this note is an extension of this fact.

Theorem 1.1. *Let X be a finite quandle and $\phi \in Z_{Q-}^2(X; \mathbb{Z})$, then for any knot K the cocycle invariant $\Phi_{\phi}^{-}(K)$ is trivial.*

Theorem 1.2. *Let X be a finite quandle and $\phi \in Z_{Q+}^2(X; \mathbb{Z})$, then for any knot K the cocycle invariant $\Phi_{\phi}^{+}(K)$ is trivial.*

We remark that if we replace the coefficient group \mathbb{Z} with some other coefficient groups, for example \mathbb{Z}_2 , then the positive(negative) quandle cocycle invariants need not to be trivial. The readers are referred to [3] for some nontrivial cocycle invariants of trefoil and the figure eight knot. On the other hand, if we replace the knot K with a link which has at least two components, then even using the trivial quandle with two elements one can prove that the negative quandle cocycle invariants contain the information of linking numbers. For the positive quandle cocycle invariant, one can even use it to distinguish the Borromean ring from the 3-component trivial link. See [3] and [4] for more details.

The remainder of this note is arranged as follows. Section 2 contains the basic definitions of quandle, positive(negative) quandle (co)homology groups and the cocycle invariants. Section 3 and section 4 are devoted to the proof of Theorem 1.1 and Theorem 1.2 respectively.

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2 Quandle homology and cocycle invariants

A *quandle* $(X, *)$, is a set X with a binary operation $(a, b) \rightarrow a * b$ satisfying the following axioms:

1. For any $a \in X$, $a * a = a$.
2. For any $b, c \in X$, there exists a unique $a \in X$ such that $a * b = c$.
3. For any $a, b, c \in X$, $(a * b) * c = (a * c) * (b * c)$.

Note that with a given binary operation $*$ we can define the dual operation $*^{-1}$ by

$$c *^{-1} b = a \text{ if } a * b = c.$$

It is easy to observe that $(X, *^{-1})$ is also a quandle, called the *dual quandle* of $(X, *)$. For convenience we simply denote $(X, *)$ by X . If X satisfies the second and the third axioms, then we name it a *rack*. Next we list some examples of quandle:

- Let $T_n = \{a_1, \dots, a_n\}$ and $a_i * a_j = a_i$, we say T_n is the trivial quandle with n elements;
- Let $D_n = \{0, 1, \dots, n-1\}$ and $i * j = 2j - i \pmod{n}$, we say D_n is the dihedral quandle of order n ;
- Let X be a conjugacy class of a group G and $a * b = b^{-1}ab$, we say X is a conjugation quandle.

Let X be a quandle and a, b two elements of X . We say a and b are of the same *orbit* if b can be obtained from a by some right translations, i.e. there exist some elements $\{a_1, \dots, a_n\} \subset X$ such that

$$b = (\dots((a *^{\varepsilon_1} a_1) *^{\varepsilon_2} a_2) \dots) *^{\varepsilon_n} a_n,$$

where $\varepsilon_i \in \{\pm 1\}$. The orbit set of X is denoted by $\text{Orb}(X)$, and the orbit that contains a is denoted by $\text{Orb}(a)$. Obviously $\text{Orb}(a)$ is a subquandle of X . If X has only one orbit then we say X is *connected*.

Note that the axioms for a quandle correspond to the three Reidemeister moves respectively. Hence by assigning each arc of a knot diagram with an element of X , such that the condition in Figure 1 is satisfied, then the number of colorings is a knot invariant, known as the quandle coloring invariant. Here by an arc we mean a part of the diagram from an undercrossing to the next undercrossing. In particular when $X = D_n$ this is exactly the well known Fox n -coloring invariant [8]. For a fixed finite quandle X , we use $\text{Col}_X(K)$ to denote the quandle coloring invariant of K .

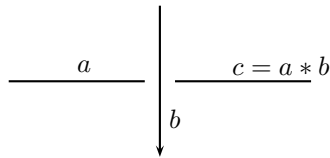


Figure 1: The coloring rule

In order to generalize the quandle coloring invariant, for a fixed coloring one can associate each crossing point with a (Boltzmann) weight such that the signed product of these weights are independent of the choice of the knot diagram. For this purpose we need to introduce the quandle (co)homology theory.

Let X be a finite quandle and $C_n^R(X)$ the free abelian group generated by n -tuples (a_1, \dots, a_n) , here each a_i is an element of X . Let us consider the following two homomorphisms from $C_n^R(X)$ to $C_{n-1}^R(X)$:

$$d_1(a_1, \dots, a_n) = \sum_{i=1}^n (-1)^i (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n) \quad (n \geq 2)$$

$$d_2(a_1, \dots, a_n) = \sum_{i=1}^n (-1)^i (a_1 * a_i, \dots, a_{i-1} * a_i, a_{i+1}, \dots, a_n) \quad (n \geq 2)$$

$$d_i(a_1, \dots, a_n) = 0 \quad (n \leq 2, i = 1, 2)$$

It is routine to check that $d_1^2 = d_2^2 = d_1 d_2 + d_2 d_1 = 0$, it follows that we have two chain complexes $\{C_n^R(X), \partial^+ = d_1 + d_2\}$ and $\{C_n^R(X), \partial^- = d_1 - d_2\}$. Define $C_n^D(X)$ ($n \geq 2$) to be the free abelian group generated by n -tuples (a_1, \dots, a_n) with $a_i = a_{i+1}$ for some $1 \leq i \leq n-1$. When $n \leq 1$ we define $C_n^D(X) = 0$. It is not difficult to check that $\{C_n^D(X), d_i\}$ is a sub-complex of $\{C_n^R(X), d_i\}$ ($i = 1, 2$), from this we conclude that $\{C_n^D(X), \partial^\pm\}$ is a sub-complex of $\{C_n^R(X), \partial^\pm\}$. Let $C_n^Q(X) = C_n^R(X)/C_n^D(X)$ and G an abelian group, then we can define the chain complexes and cochain complexes ($W \in \{R, D, Q\}$)

- $C_*^{W\pm}(X; G) = \{C_*^W(X), \partial^\pm\} \otimes G, \quad \partial^\pm = \partial^\pm \otimes \text{id};$
- $C_{W\pm}^*(X; G) = \text{Hom}(\{C_*^W(X), \partial^\pm\}, G), \quad \delta_\pm = \text{Hom}(\partial^\pm, \text{id}).$

Now we define the *positive quandle (co)homology groups* and *negative quandle (co)homology groups* of X with coefficient G to be the (co)homology groups of $C_*^{Q\pm}(X; G)$ ($C_{Q\pm}^*(X; G)$). The positive(negative) rack (co)homology groups and positive(negative) degeneration (co)homology groups can be defined in an analogous way. We remark that the negative quandle (co)homology groups are exactly the well studied quandle (co)homology groups defined by J. S. Carter et al in [3]. In order to distinguish it from the positive quandle (co)homology groups we will call it negative quandle (co)homology groups throughout this note.

Now we turn to the definition of cocycle invariants. Let K be a knot diagram and X a fixed finite quandle, we use ρ to denote a coloring of K by X . It is well known that the regions of $R^2 - K$ have a checkerboard fashion coloring. Without loss of generality we assume the unbounded region has the white color. Then we can assign each crossing τ of K with a sign $\epsilon(\tau)$ according to Figure 2.



Figure 2: The signs of crossings

Assume the over-arc and under-arcs of τ are colored by b and $a, a * b$ respectively, see Figure 1. Then with a given negative quandle 2-cocycle ϕ we can define a weight $W_\phi(\tau, \rho)$ associate to τ as

$$W_\phi^-(\tau, \rho) = \phi(a, b)^{w(\tau)},$$

here $w(\tau)$ denotes the writhe of the crossing. The *negative quandle 2-cocycle invariant* of K is defined to be

$$\Phi_\phi^-(K) = \sum_\rho \prod_\tau W_\phi^-(\tau, \rho) \in \mathbb{Z}G,$$

where ρ runs all colorings of K by X and τ runs all crossing points of the diagram. If ϕ is a positive quandle 2-cocycle, we need to make a little modification on the definition of $W_\phi^-(\tau, \rho)$. By replacing $w(\tau)$ with $\epsilon(\tau)$, we define

$$W_\phi^+(\tau, \rho) = \phi(a, b)^{\epsilon(\tau)},$$

and the *positive quandle 2-cocycle invariant* of K can be defined by

$$\Phi_\phi^+(K) = \sum_\rho \prod_\tau W_\phi^+(\tau, \rho) \in \mathbb{Z}G,$$

similarly here ρ runs all colorings of K by X and τ runs all crossing points of K .

Theorem 2.1. [3] *The negative quandle 2-cocycle invariant $\Phi_\phi^-(K)$ is invariant under Reidemeister moves. Moreover if a pair of negative quandle 2-cocycles ϕ_1 and ϕ_2 are cohomologous, then $\Phi_{\phi_1}^-(K) = \Phi_{\phi_2}^-(K)$.*

Theorem 2.2. [4] *The positive quandle 2-cocycle invariant $\Phi_\phi^+(K)$ is invariant under Reidemeister moves. Moreover if a pair of positive quandle 2-cocycles ϕ_1 and ϕ_2 are cohomologous, then $\Phi_{\phi_1}^+(K) = \Phi_{\phi_2}^+(K)$.*

We remark that with a positive(negative) quandle 3-cocycle one can also define a state-sum invariant for knotted surfaces [3, 4], but we will not discuss it in this note. On the other hand, in a recent paper [10] of S. Kamada, V. Lebed and K. Tanaka, by investigating the relation between the Alexander numbering and $\epsilon(\tau)$ they studied the relation between positive quandle 2-cocycle invariants and the twisted quandle cocycle invariants, which was proposed by J. S. Carter, M. Elhamdadi and M. Saito in [2]. See [10] for more details.

3 The proof of Theorem 1.1

From now on the coefficient group G is supposed to be \mathbb{Z} , and as usual we will write $H_{Q^\pm}^*(X, \mathbb{Z})$ simply by $H_{Q^\pm}^*(X)$. We begin the proof with a simple lemma.

Lemma 3.1. *If ϕ has finite order in $H_{Q_-}^2(X)$, then $\Phi_\phi^-(K)$ is trivial for any knot K .*

Proof. Assume $k\phi = 0 \in H_{Q_-}^2(X)$, then by Theorem 2.1 we have $\prod_\tau W_{k\phi}^-(\tau, \rho) = 0$ for each coloring ρ . In other words, $\prod_\tau k\phi(a, b)^{w(\tau)} = k(\prod_\tau \phi(a, b)^{w(\tau)}) = 0$. Since we are working with coefficient \mathbb{Z} , it implies $\prod_\tau \phi(a, b)^{w(\tau)} = 0$. \square

Lemma 3.1 tells us that it suffices to consider the free part of $H_{Q_-}^2(X)$. The following key lemma mainly follows from P. Etingof and M. Graña's result [5].

Lemma 3.2. *If X is a connected quandle, then $\Phi_\phi^-(K)$ is trivial for all knots K .*

Proof. According to the definition of negative quandle cohomology group, we have a long exact sequence

$$\cdots \rightarrow H_n^{D-}(X) \rightarrow H_n^{R-}(X) \rightarrow H_n^{Q-}(X) \rightarrow H_{n-1}^{D-}(X) \rightarrow \cdots$$

It was proved by Litherland and Nelson [11] that this sequence is split into short exact sequences, i.e. there exists a short exact sequence

$$0 \rightarrow H_n^{D-}(X) \rightarrow H_n^{R-}(X) \rightarrow H_n^{Q-}(X) \rightarrow 0.$$

On the other hand, in [5] P. Etingof and M. Graña calculated the Betti numbers of negative rack cohomology groups of X are $\dim H_{R-}^n(X; \mathbb{Q}) = |\text{Orb}(X)|^n$. In particular, when X is connected we have $\dim H_{R-}^2(X; \mathbb{Q}) = 1$.

Now we turn to $H_2^{D-}(X)$. Since

$$\begin{aligned} \partial^-(a, a) &= -((a) - (a)) + ((a) - (a * a)) = 0 \\ \partial^-(a, a, b) &= -(a, b) + (a, b) + (a, b) - (a * a, b) - (a, a) + (a * b, a * b) = -(a, a) + (a * b, a * b) \\ \partial^-(a, b, b) &= -(b, b) + (b, b) + (a, b) - (a * b, b) - (a, b) + (a * b, b * b) = 0, \end{aligned}$$

it follows that $H_2^{D-}(X) = \mathbb{Z}[\text{Orb}(X)]$, the free abelian group generated by the $\text{Orb}(X)$. Recall that X is connected, hence $H_2^{D-}(X) = \mathbb{Z}$.

According to the discussion above, together with the universal coefficient theorem [1], we conclude that $H_{Q_-}^2(X; \mathbb{Q}) = 0$. The result follows. \square

Now we give the proof of Theorem 1.1.

Proof. Let $\rho : Q(K) \rightarrow X$ denote a coloring of K by X , it is sufficient to show that $\prod_{\tau} W_{\phi}^{-}(\tau, \rho) = 0$. Since K has only one component, it is obvious that $\rho(Q(K))$ belong to the same orbit of X . Choose an element $a \in \rho(Q(K))$, since $\rho(Q(K)) \subset \text{Orb}(a)$, therefore ρ represents a coloring of K by $\text{Orb}(a)$. Let ϕ be a negative quandle 2-cocycle of X , then it satisfies the negative quandle 2-cocycle condition

$$\phi(x, z) - \phi(x * y, z) - \phi(x, y) + \phi(x * z, y * z) = 0 \text{ for } \forall x, y, z \in X.$$

Now we define a map $\phi' : \text{Orb}(a) \times \text{Orb}(a) \rightarrow \mathbb{Z}$ by

$$\phi'(x, y) = \phi(x, y) \text{ for } \forall x, y \in \text{Orb}(a).$$

Obviously $\phi' \in Z_{Q-}^2(\text{Orb}(a))$. Then we have

$$\prod_{\tau} W_{\phi}^{-}(\tau, \rho) = \prod_{\tau} W_{\phi'}^{-}(\tau, \rho) = 0,$$

the last equality comes from the fact that $\text{Orb}(a)$ is connected and Lemma 3.2. The proof is finished. \square

4 The proof of Theorem 1.2

In this section we give the proof of Theorem 1.2, which is quite different from that of Theorem 1.1.

Lemma 4.1. *Let X be a finite quandle, a an element of X and $\phi \in Z_{Q+}^2(X)$, suppose ρ is a coloring of K by X , then $\rho * a$ is also a coloring of K by X . Here $\rho * a$ denotes the coloring which assigns $x * a$ to an arc if ρ assigns x to it. Moreover we have $\prod_{\tau} W_{\phi}^{+}(\tau, \rho) + \prod_{\tau} W_{\phi}^{+}(\tau, \rho * a) = 0$.*

Proof. $\rho * a$ is a coloring follows directly from the third axiom of quandle.

Now we show that $\prod_{\tau} W_{\phi}^{+}(\tau, \rho) + \prod_{\tau} W_{\phi}^{+}(\tau, \rho * a) = 0$. Because $\phi \in Z_{Q+}^2(X)$, thus ϕ satisfies the positive quandle 2-cocycle condition

$$-\phi(y, z) - \phi(y, z) + \phi(x, z) + \phi(x * y, z) - \phi(x, y) - \phi(x * z, y * z) = 0 \text{ for } \forall x, y, z \in X.$$

It follows that

$$\phi(x, y) + \phi(x * z, y * z) = -\phi(y, z) - \phi(y, z) + \phi(x, z) + \phi(x * y, z).$$

Replacing z with a we have

$$\phi(x, y) + \phi(x * a, y * a) = -\phi(y, a) - \phi(y, a) + \phi(x, a) + \phi(x * y, a).$$

One computes

$$\begin{aligned} & \prod_{\tau} W_{\phi}^{+}(\tau, \rho) + \prod_{\tau} W_{\phi}^{+}(\tau, \rho * a) \\ &= \prod_{\tau} \phi(x, y)^{\epsilon(\tau)} + \prod_{\tau} \phi(x * a, y * a)^{\epsilon(\tau)} \\ &= \prod_{\tau} (\phi(x, y) + \phi(x * a, y * a))^{\epsilon(\tau)} \\ &= \prod_{\tau} (-\phi(y, a) - \phi(y, a) + \phi(x, a) + \phi(x * y, a))^{\epsilon(\tau)} \end{aligned}$$

The second equality holds as we are working with coefficient \mathbb{Z} , for the same reason $\prod_{\tau} (-\phi(y, a) - \phi(y, a) + \phi(x, a) + \phi(x * y, a))^{\epsilon(\tau)}$ can be rewritten as $\sum_{\tau} \epsilon(\tau) (-\phi(y, a) - \phi(y, a) + \phi(x, a) + \phi(x * y, a))$.

In order to finish the proof we need to show that $\sum_{\tau} \epsilon(\tau) (-\phi(y, a) - \phi(y, a) + \phi(x, a) + \phi(x * y, a)) = 0$. Notice that if the over-arc and under-arcs of τ are colored by y and $x, x * y$ respectively, then the contribution that comes from τ equals $\epsilon(\tau) (-\phi(y, a) - \phi(y, a) + \phi(x, a) + \phi(x * y, a))$. We can divide this contribution into three parts: $-2\epsilon(\tau)\phi(y, a)$, $\epsilon(\tau)\phi(x, a)$, $\epsilon(\tau)\phi(x * y, a)$, and assign them to the over-arc and two under-arcs respectively. If the diagram is alternating, then all $\epsilon(\tau)$ are the same. In this case it is not difficult to observe that the contributions from all crossings will cancel out. The proof of the non-alternating case is analogous to the alternating case. In fact it suffices to notice that if an arc crosses several crossings as the over-arc, then the sign ϵ of these crossings are alternating. It follows that in this general case we still have $\sum_{\tau} \epsilon(\tau) (-\phi(y, a) - \phi(y, a) + \phi(x, a) + \phi(x * y, a)) = 0$. \square

Lemma 4.2. $\prod_{\tau} W_{\phi}^{+}(\tau, \rho) = \prod_{\tau} W_{\phi}^{+}(\tau, \rho * a)$.

Proof. Let K be a knot diagram. If we place a unknot diagram U near K which has no intersection with K then we obtain a two-component link diagram $L = K \cup U$. Obviously K and L have the same positive quandle 2-cocycle invariants if U is always colored by a . Let us move K into the circle U via some second and third Reidemeister moves, when K meets U during the move we always put U over K . Now we obtain a new link diagram L' . By checking the variations of coloring under Reidemeister moves one will find that if an arc of L is colored by x then it will be colored by $x * a$ in L' . See Figure 3.

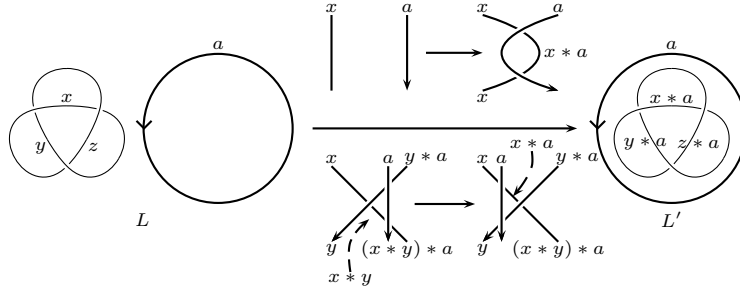


Figure 3: Transformation between L and L'

Let K be a knot diagram and ρ a coloring of K by X . If K' can be obtained from K by a sequence of Reidemeister moves then these Reidemeister moves will induce a coloring ρ' on K' . It was proved in [4] that ρ and ρ' offer the same contribution to the positive quandle 2-cocycle invariants. Together with the discussion above we conclude that $\prod_{\tau} W_{\phi}^{+}(\tau, \rho) = \prod_{\tau} W_{\phi}^{+}(\tau, \rho * a)$. \square

Theorem 1.2 follows directly from Lemma 4.1 and Lemma 4.2.

Remark There is another proof of Lemma 4.2 without needing to introduce the unknot diagram U [13]. Choose an arc λ of K , without loss of generality we assume that λ is colored by a . Let us take the first Reidemeister move Ω_1 on λ and slide the remainder of K through the small loop created by the Reidemeister move. After taking Ω_1^{-1} on λ we obtain the coloring $\rho * a$ on K . On the other hand we want to remark that if we replace the knot K by a link L and replace the coefficient group \mathbb{Z} by an abelian group without 2-torsion, Theorem 1.2 still holds.

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